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School of Mathematics

# Student approaches to generating mathematical examples

Comparing e-assessment and paper-based tasks

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 @georgekinnear

## Example generation: background



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## Comparing e-assessment with pen and paper



Paola Iannone  
University of Edinburgh



Ben Davies  
University of Southampton



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# Example generation: background



# Example generation

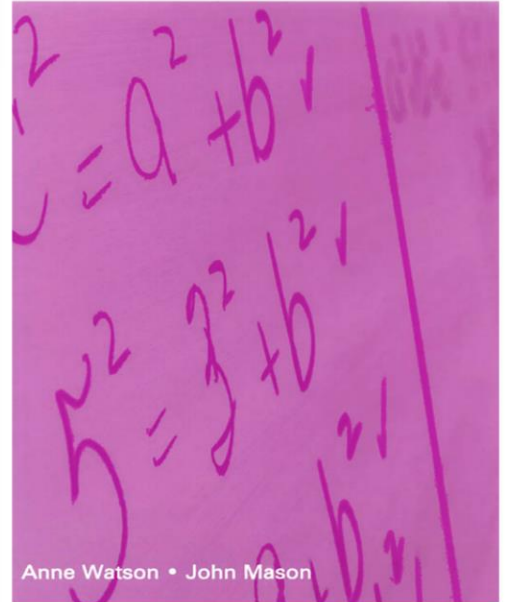
“an important and effective pedagogical strategy whose potential is rarely exploited yet which promotes active engagement in mathematics” (p. x)

- Sketch a cubic
- ...which does not go through the origin
- ...and which is monotonic
- ...and for which the inflection slope is negative



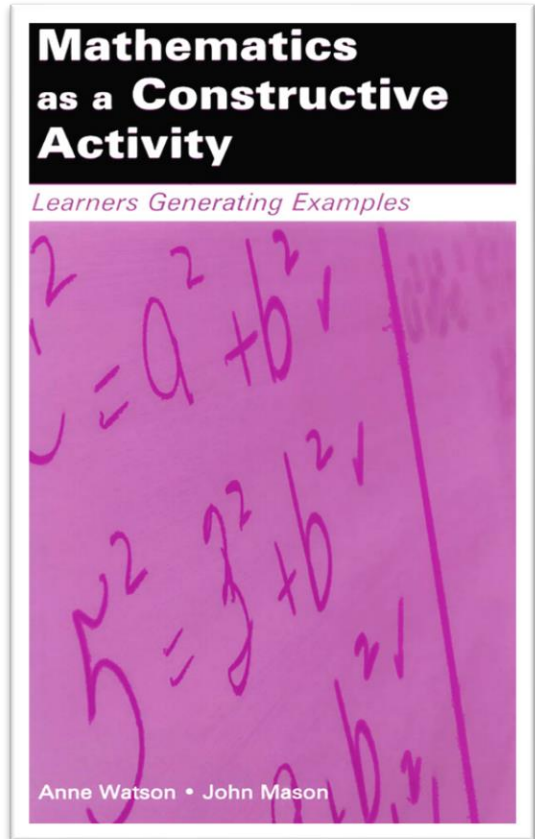
## Mathematics as a Constructive Activity

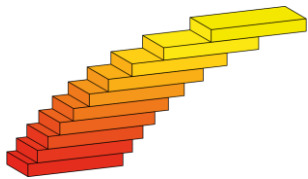
*Learners Generating Examples*



# Example generation

- Suggested prompts:
- Make up an example.
- Make up an example with some constraints.
- Add constraints sequentially.
- Make up another or more like or unlike this.
- ...






# STACK e-assessment

- Randomisation
- Algebraic input
- Robust grading

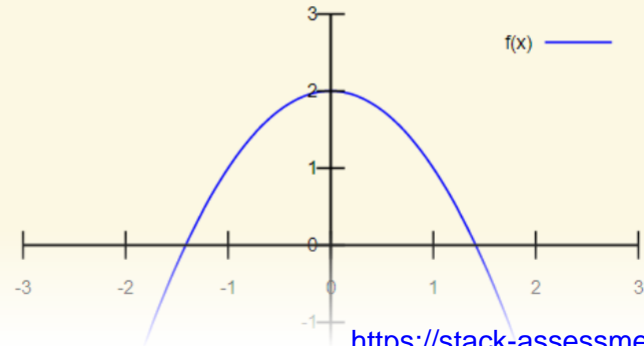
Give an example of a function with y-intercept 2 and x-intercept 1.

$f(x) =$

Check

 Your answer is partially correct.

Your function does not have x-intercept 1. Instead, it has x-intercept(s)  $x = -\sqrt{2}$ ,  $x = \sqrt{2}$  as shown below.



<https://stack-assessment.org/About/>



# Quadratic intersections

For each case below, type in a quadratic, e.g.  $2x^2+3x+1$ , whose graph has exactly the given number of intersections with  $y = x^2$ . If it is not possible, then enter **none**.

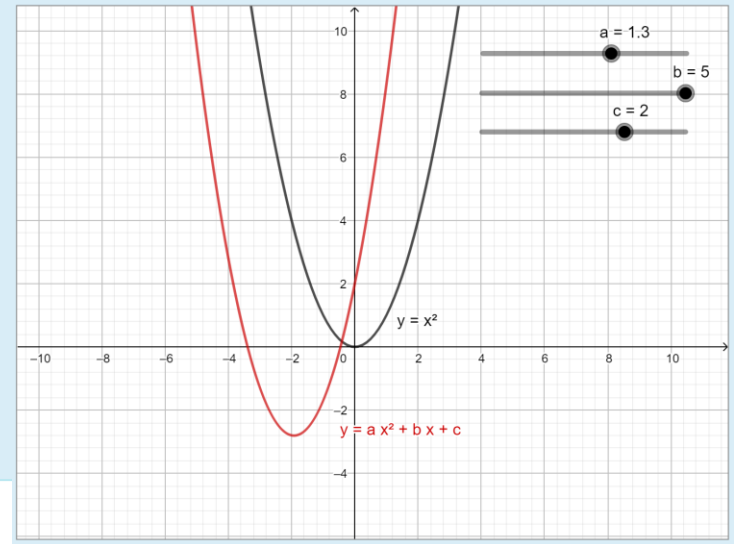
No intersection:  $y =$

1 intersection:  $y =$

2 intersections:  $y =$

3 intersections:  $y =$

Check



# Linear algebra

For each box, give an example of a matrix which satisfies the conditions given.

If no such matrix exists, leave blank.

	Rank		
	1	2	3
diagonal	$\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$	$\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$	$\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$
non-diagonal	$\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$	$\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$	$\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$





# Linear algebra

For each of the statement below, select whether it is true or false, and give either a proof or counterexample as appropriate.

Suppose  $w, x, y$  are vectors in  $\mathbb{R}^3$ .

(a) If  $\{w, x\}$  is independent, then  $\{w, x, x - w\}$  is independent.

This statement is

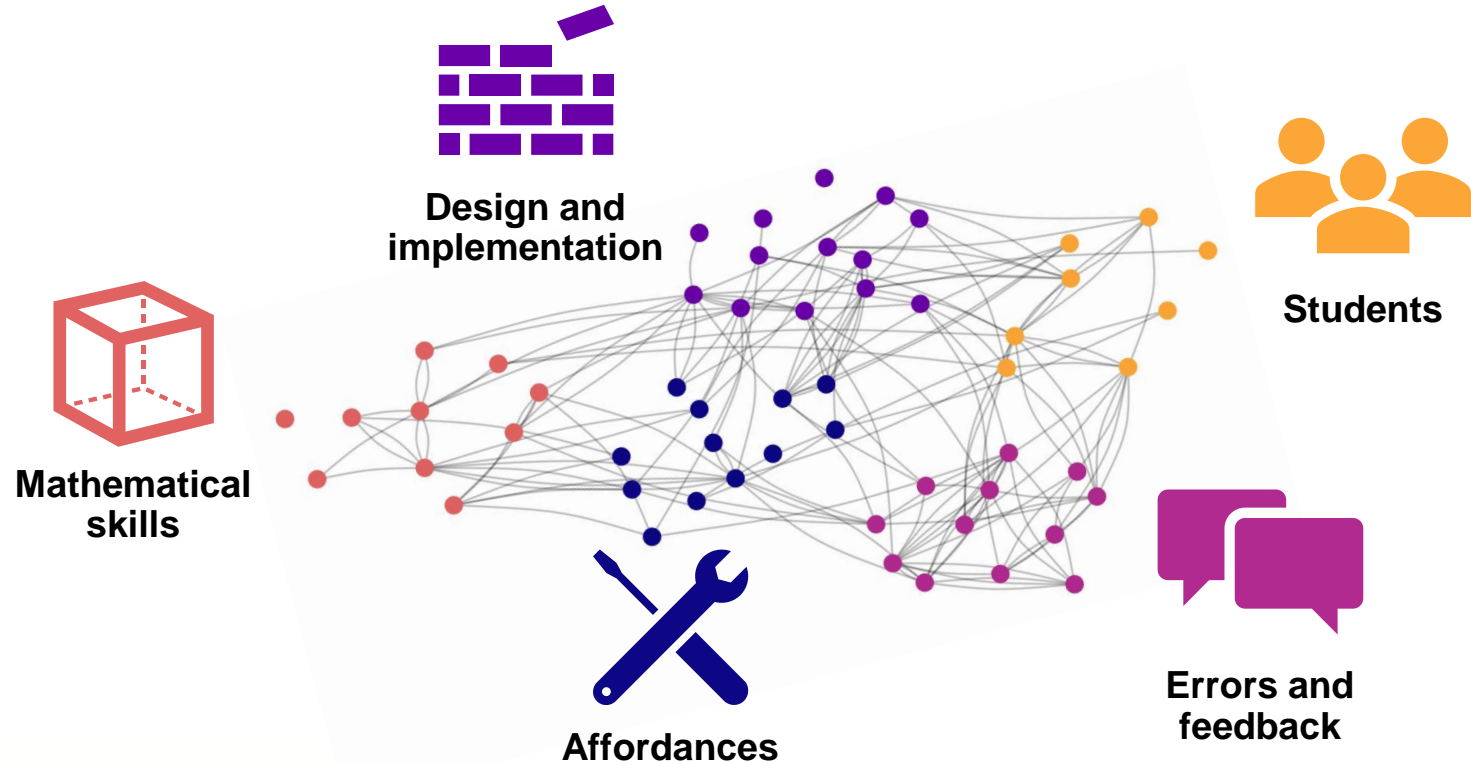
Counterexample:

$$w = \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}, x = \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}$$

(b) If  $\{w, x, y\}$  is independent, then  $\{w, x + y\}$  is independent.



# Research agenda



# Research agenda

Question 53

**How can e-assessments be designed to expand and enrich students' example spaces?**

 [George Kinnear](#)

 [Colin Foster](#)

Learning about concepts by developing a rich example space has been suggested as particularly currently about de

Question 54

**To what extent can e-assessments meaningfully judge student responses to example generation tasks?**

 [George Kinnear](#)

 [Chris Sangwin](#)

 [Tim Lowe](#)

 [Peter Rowlett](#)

What r

Watson a  
students  
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examples

Some CAA systems appear well-placed to facilitate example generation tasks, as they can make use of sophisticated computer algebra systems to check the properties of student examples and give feedback - however this relies on the properties being ones that the system can evaluate.

**What motivates this question?**

One diffi  
mathema  
such as S  
if not, pe

Learner-generated example tasks have been suggested as an effective way to encourage students to engage with new concepts (Watson & Mason, 2006). Checking students' answers and giving feedback can be a difficult task for teachers, particularly with large groups of students. Thus, e-assessment could play a powerful role in providing students

Question 55

**How does the use of e-assessment impact students' example generation strategies and success, relative to the same tasks on paper or orally?**

 [George Kinnear](#)

**What motivates this question?**

Students' strategies have been studied in previous work (e.g. Iannone et al., 2011) but computer-aided assessment brings additional constraints that warrant further investigation. For instance, students may immediately be able to sketch an example with the required properties, or orally describe the relevant features, but may struggle to formulate this as an algebraic expression that is suitable for entry into the CAA system. As a specific example, they may wish to use a piecewise function but may not know the notation to use (if it is even possible in a given CAA system).

**What might an answer look like?**

This would likely require in-depth qualitative investigation of students' strategies, e.g. through observations or clinical interviews (as in Iannone et al., 2011).

The mathematical topic is likely to be an important factor, as suggested by the example of

<https://maths.github.io/e-assessment-research-agenda>

# Studying example generation

G2

Give the formula for the  $n$ th term of a sequence  $u_1, u_2, u_3, \dots$  which is:

- decreasing

$$u_n = \text{[input box]}$$

- decreasing and bounded below

$$u_n = \text{[input box]}$$

- decreasing, bounded below, and has  $u_4 = 1$

$$u_n = \text{[input box]}$$

G4

Drag the points  $u_1, \dots, u_8$  so that they show the first 8 terms of an increasing sequence.

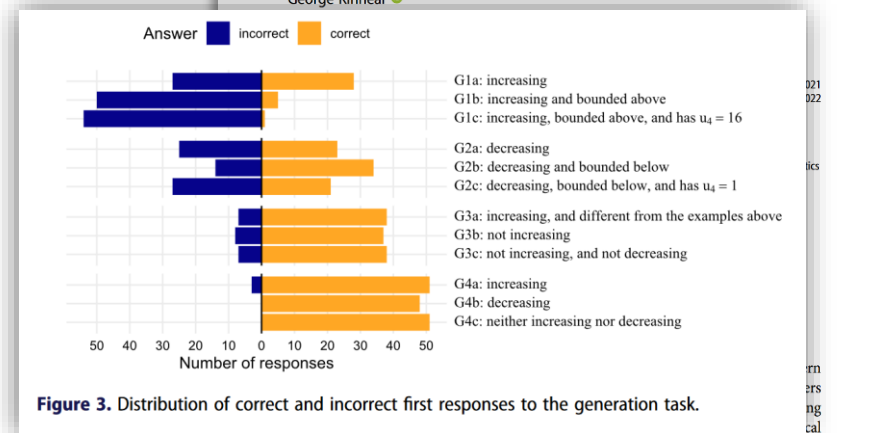
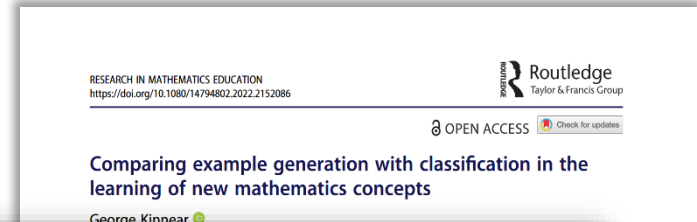
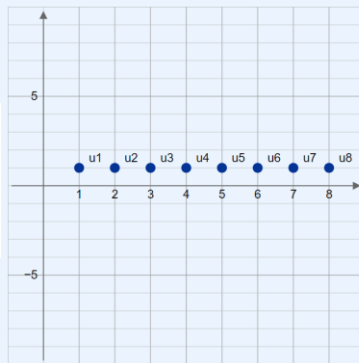


Figure 3. Distribution of correct and incorrect first responses to the generation task.

support for this approach is lacking. Jannone et al. (2011) noted that “empirical support for the hypothesis that example generation aids in concept understanding appears to come from just seven students” (p. 10), and their own experiment (with 53 students) found no difference in performance on a proof production task for students who were prompted



# Comparing e-assessment with pen and paper



Paola Iannone  
University of Edinburgh



Ben Davies  
University of Southampton



How do students respond to  
e-assessment versions of  
example-generation tasks?



# Give an example...

Give an example of a function  $f : [0, 1] \rightarrow [0, 1]$  with image  $[0, \frac{1}{2}]$ .



# Study 1: Analysing responses

- Embedded in a first-year undergraduate course, *Proofs and Problem Solving*

## Written

- 2020/21
- N=333
- Week 8 homework  
(Best 8 out of 10 count for 25% of course)

## STACK

- 2019/20
- N=322
- Week 8 “reading test”  
(Best 8 out of 10 count for 5% of course)





# Written task

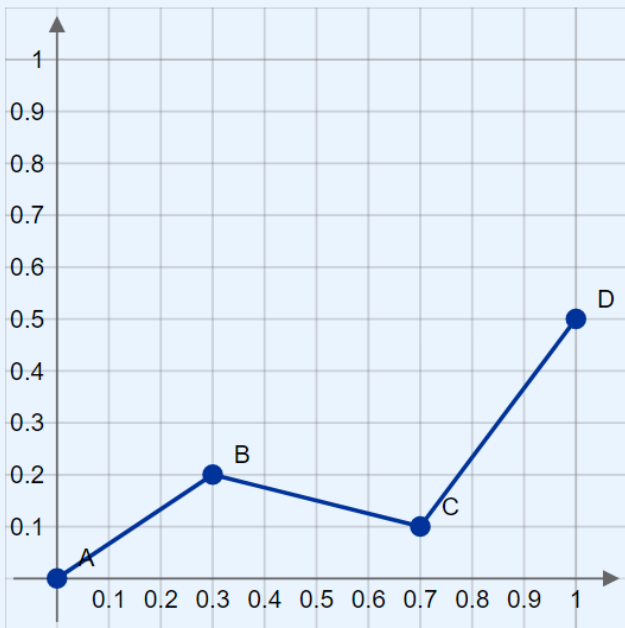
1. (a) In each case, draw the graph of a function with the given properties, and label important points:

- $f_1 : [0, 1] \rightarrow [0, 1]$  has image  $[0, \frac{1}{2}]$ .
- $f_2 : [0, 1] \rightarrow [0, 1]$  has image  $[0, \frac{1}{2}]$  and is not injective.
- $f_3 : [0, 1] \rightarrow [0, 1]$  is surjective and not injective.
- $f_4 : [0, 1] \rightarrow [0, 1]$  is injective, not surjective, and passes through  $(0.2, 0.8)$  and  $(0.5, 0.5)$ .



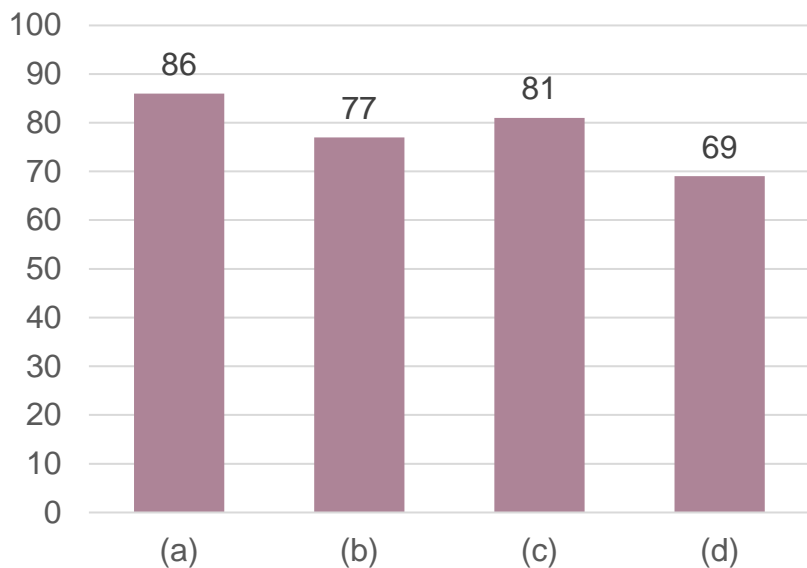
# STACK version

Drag the points so that the diagram shows the graph of a function  $f : [0, 1] \rightarrow [0, 1]$  with image  $[0, \frac{1}{2}]$ .

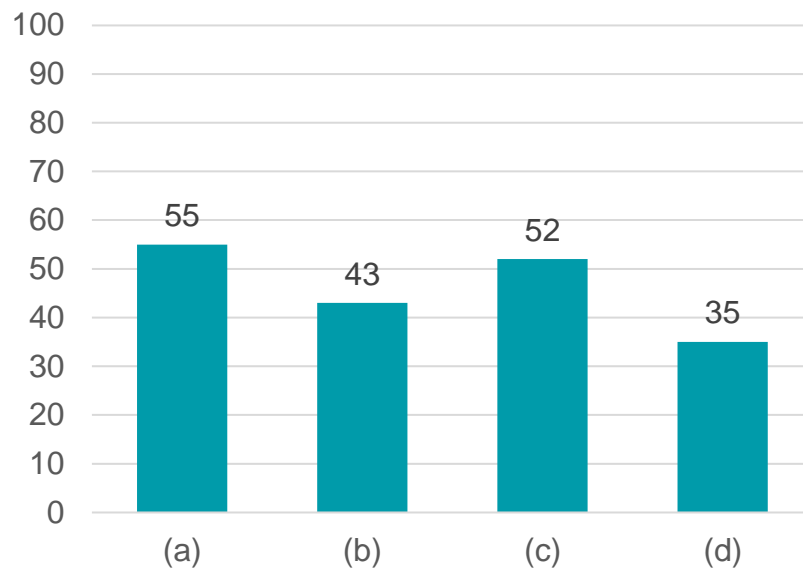


# Comparing success rates

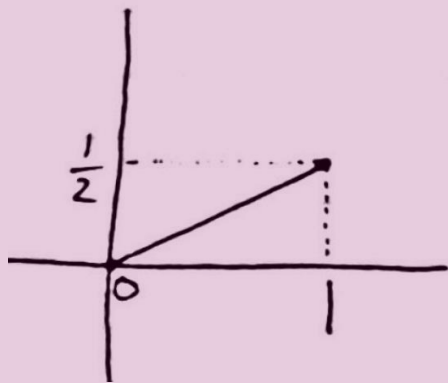
## Written



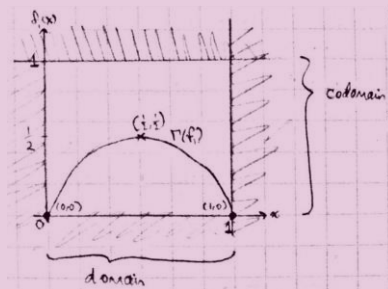
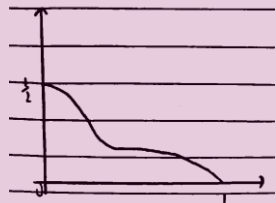
## STACK



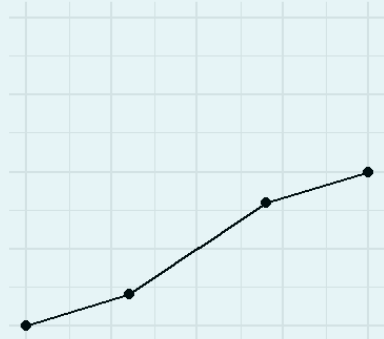
**Task A:**  $f_1 : [0, 1] \rightarrow [0, 1]$  has image  $[0, \frac{1}{2}]$



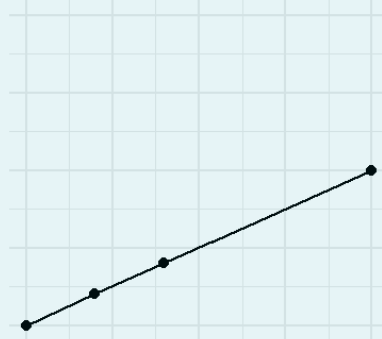
**90%**  
(258 of 286  
correct  
responses)



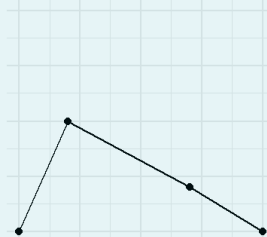
increasing (n=84)



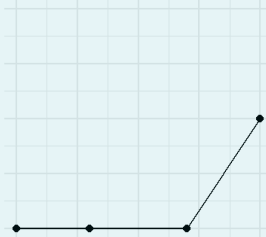
$y = x/2$  (n=53)



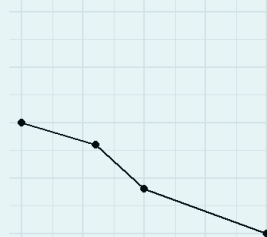
hump / inverted v (n=10)



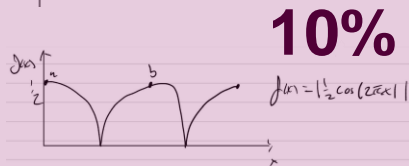
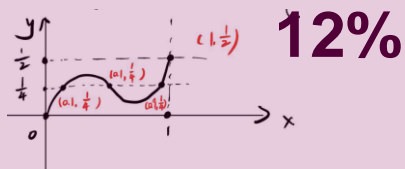
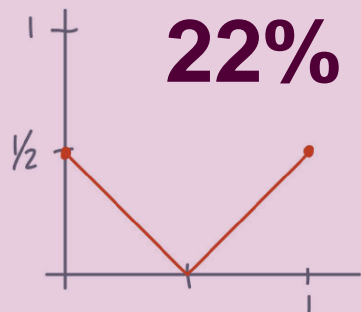
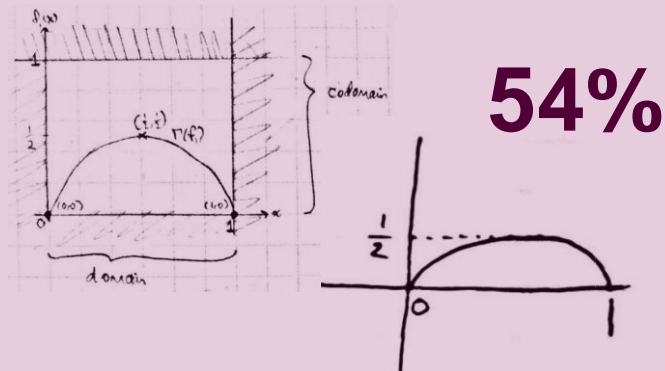
only one point moved (n=11)



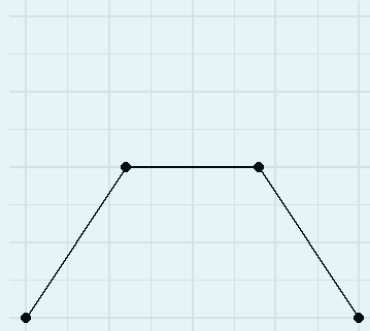
other (n=19)



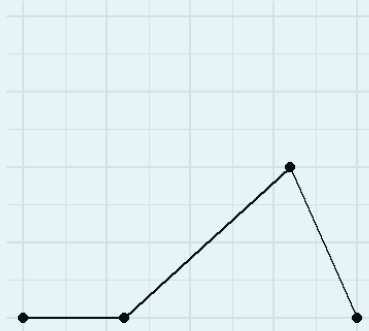
**Task B:**  $f_2 : [0, 1] \rightarrow [0, 1]$  has image  $[0, \frac{1}{2}]$  and is not injective.



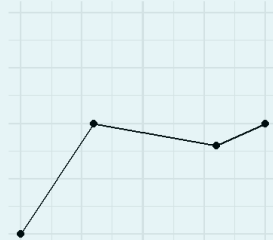
hump (n=43)



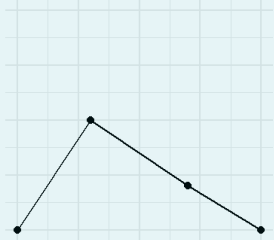
flat part (n=50)



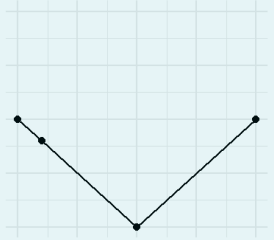
zig zag (n=25)



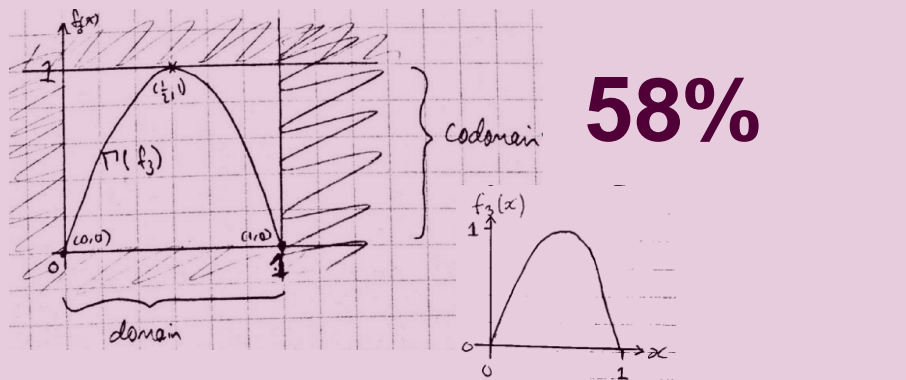
concave down (n=13)



other (n=8)

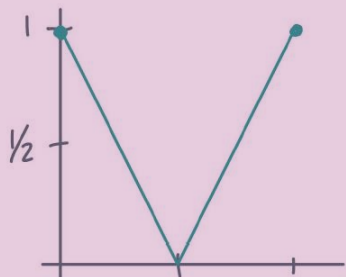


**Task C:**  $f_3 : [0, 1] \rightarrow [0, 1]$  is surjective and not injective.

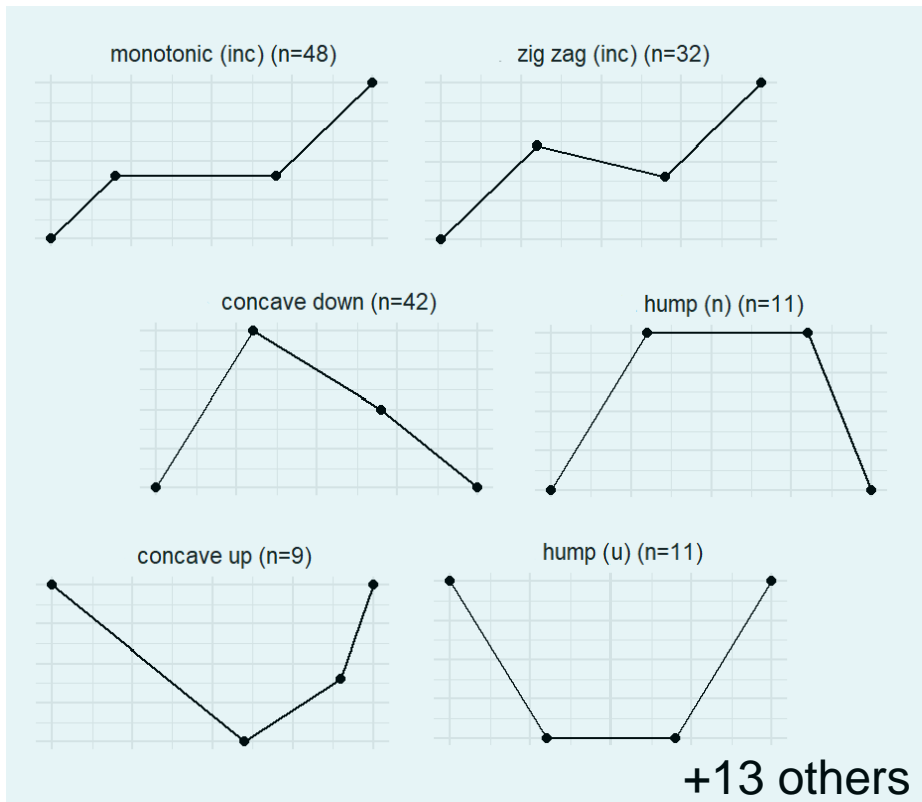
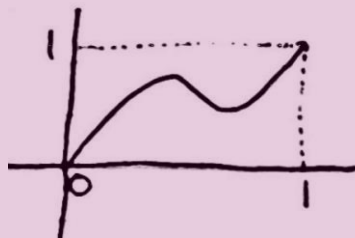


**58%**

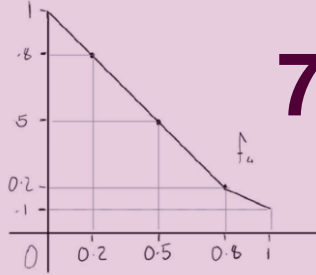
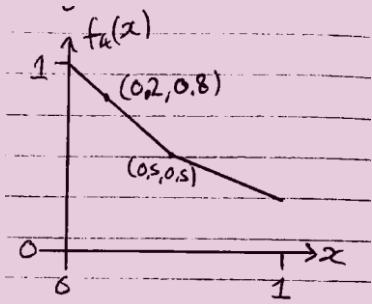
**20%**



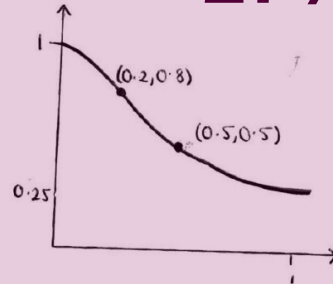
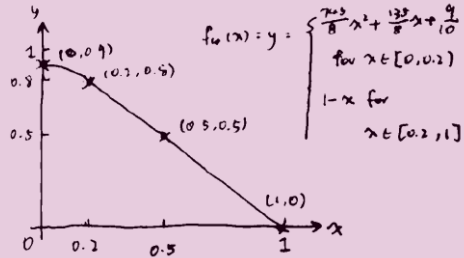
**14%**



**Task D:**  $f_4 : [0, 1] \rightarrow [0, 1]$  is injective, not surjective, and passes through  $(0.2, 0.8)$  and  $(0.5, 0.5)$ .

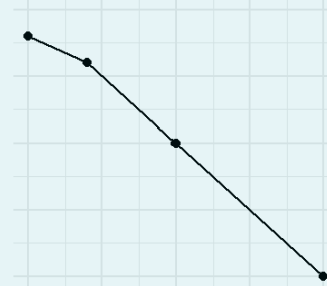


72%

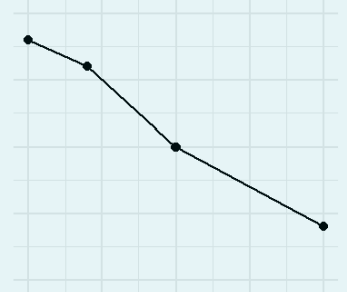


27%

top end (n=55)



both ends (n=28)



bottom end (n=24)



# Summary of findings

- Students performed better on the written task than on the e-assessment version
- Greater diversity of examples in written responses





# So what on earth is happening?

*Some conjectures...*

- ❓ Limiting the example space to piecewise functions only makes the task more difficult.
- ❓ The format of the STACK task was unfamiliar.
- ❓ Students tried harder on the handwritten task.
- ❓ Timing and assessment context...



# Study 2: Interviews with students

- Volunteers recruited from the same course in 2021/22

## Written

- 11 students
- In person, video recorded

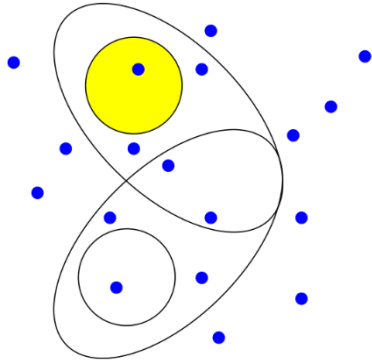
## STACK

- 8 students
- Via Teams, with shared screen recorded



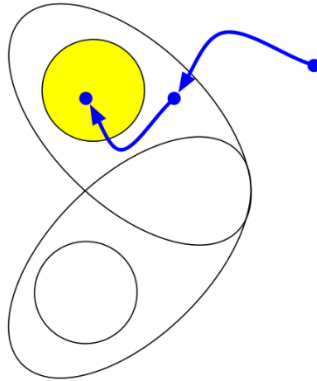
# Example generation strategies

Trial and error



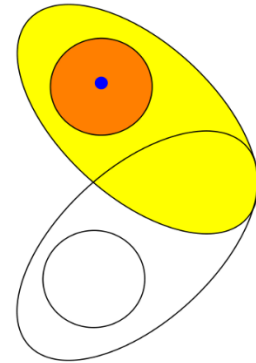
Recall or otherwise generate examples and test whether they have the desired property

Transformation



Start with an example which fits some part of the properties then make changes to bring it closer

Analysis

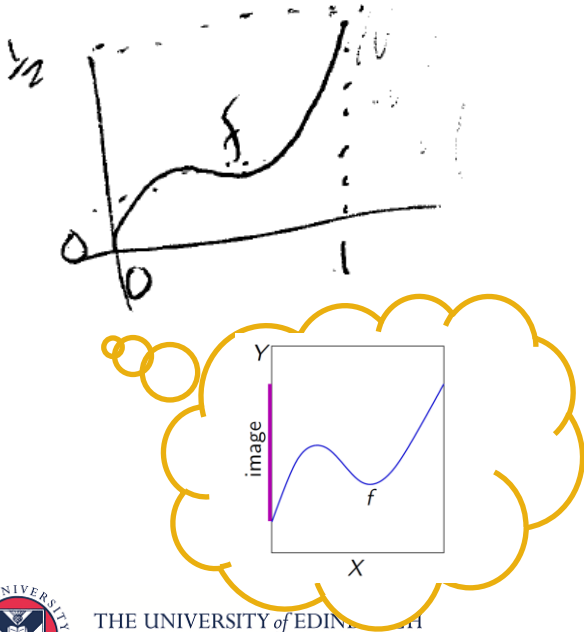


Isolate desired features and deduce the conditions they impose on examples, and gradually refine

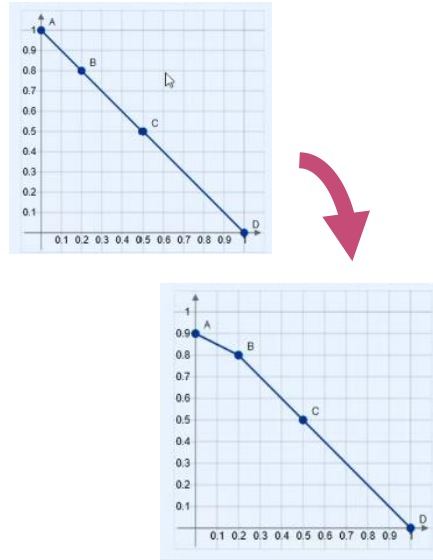


# Do the students use different strategies?

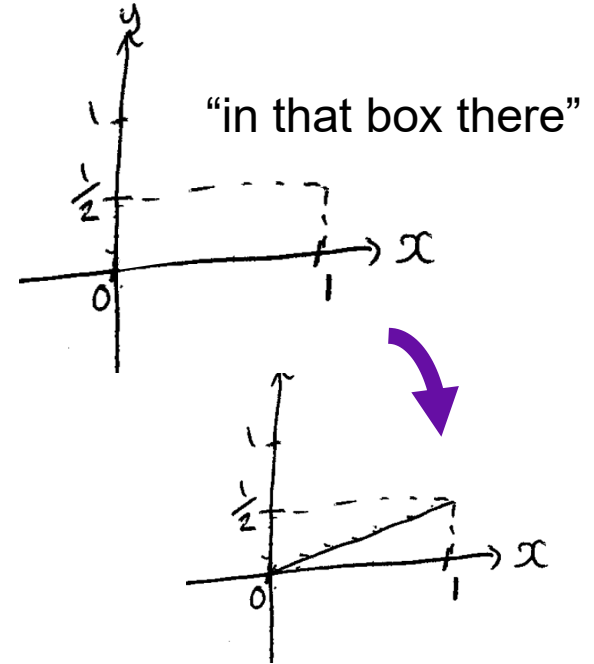
## Trial and error



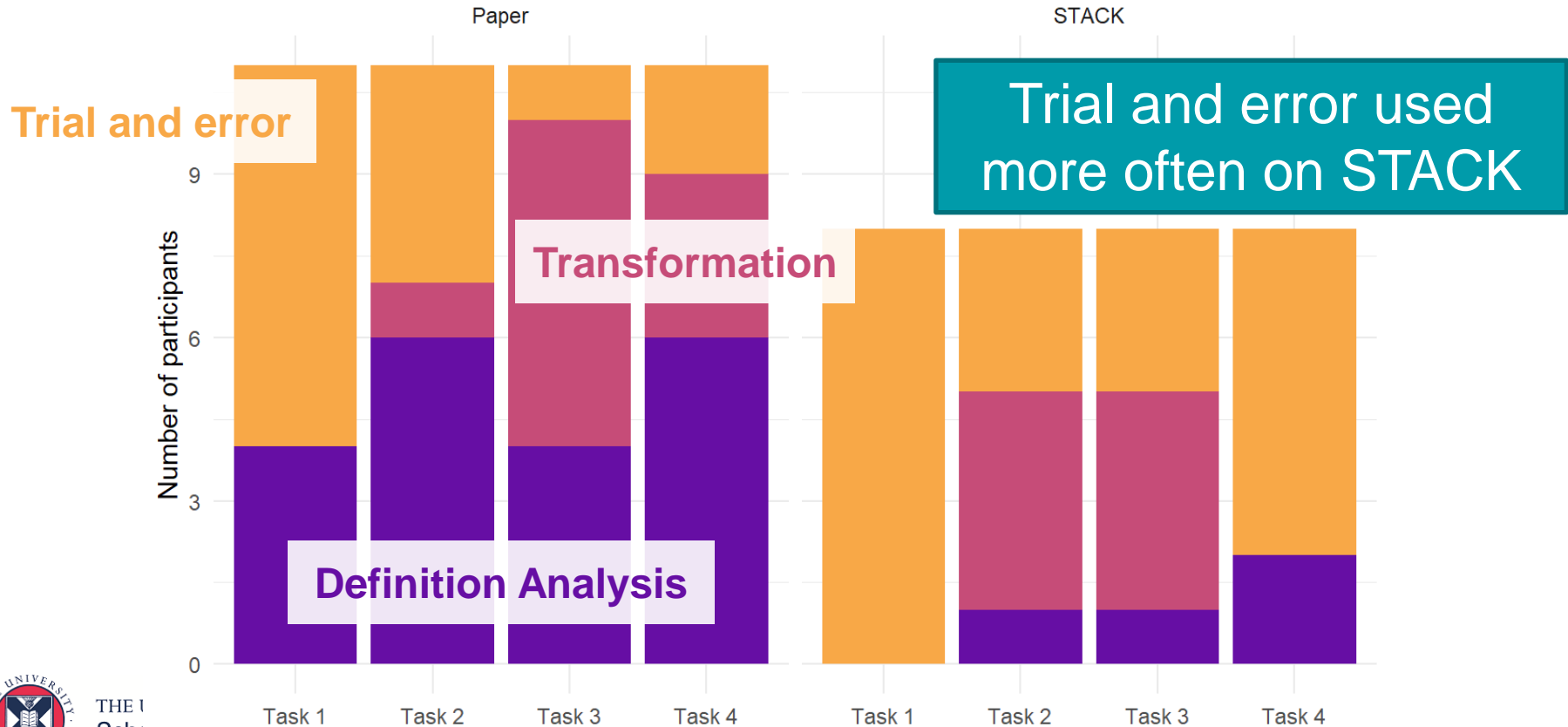
## Transformation



## Definition Analysis



# Do the students use different strategies?



# Initial thoughts

- E-assessment format adds challenge
  - Using the interface
  - Expressing answers
  - Encouraging strategies that seem to be less effective
- Some of the challenge may be helpful (e.g. drawing attention to piecewise functions)



**Thank you!**



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